Introduction to Mathematics and Modeling

lecture 6

Complex numbers

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lecture : 6
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1. Appendix A.7: complex numbers
2. Application: impedance
A complex number is a vector in $\mathbb{R}^2$.
- In stead of $\mathbb{R}^2$ we write $\mathbb{C}$.
- Rather than $x$- and $y$-axis, we call them the real axis and imaginary axis.
- The complex number $i$ is defined as $(0, 1)$. 
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The complex number $i$ is defined as $(0, 1)$.
Addition and scalar multiplication

- Addition is defined termwise: if \( z = (x, y) \) and \( w = (u, v) \), then

\[
z + w = (x + y, y + v)
\]

- Scalar multiplication is defined termwise: if \( z = (x, y) \) and \( \alpha \in \mathbb{R} \), then

\[
\alpha z = (\alpha x, \alpha y)
\]

Notebook: Sum.nb
Definition

Let \( z = (x, y) \) and \( w = (u, v) \) be two complex numbers. The **product** of \( z \) and \( w \) is defined as

\[
z w = (x u - y v, x v + y u)
\]

Examples:

- \((1, 2)(4, -1) = \)
- \((2, 0)(3, -4) = \)
- \(i^2\)

Notebook: Product.nb
Every real number $x$ is identified with the complex number $(x, 0)$.

Examples: $0 = (0, 0)$, $1 = (1, 0)$, $-1 = (-1, 0)$.

The complex numbers on the real axis behave just like the real numbers in $\mathbb{R}$:

- $x + y \rightarrow (x, 0) + (y, 0) = (x + y, 0 + 0) = (x + y, 0)$.
- $x - y \rightarrow (x, 0) - (y, 0) = (x - y, 0 - 0) = (x - y, 0)$.
- $x y \rightarrow (x, 0)(y, 0) = (xy - 0 \cdot 0, x \cdot 0 + 0 \cdot y) = (xy, 0)$. 
Real numbers are complex numbers

- By identifying $x \in \mathbb{R}$ with the complex number $(x, 0)$, we regard the points on the real axis as the real number line.

$$i^2 = -1$$

- The complex numbers are an expansion of the real numbers:

\[\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}\]
Let $z$, $w$ and $u$ be complex numbers. Define $z - w$ and $-z$ in the usual way, then

1. $z + w = w + z$

2. $z + w + u = z + (w + u) = (z + w) + u$

3. $z + 0 = z$

4. $-z = (-1)z$

5. $z - w = z + (-w)$

6. $z - z = 0$

7. $zw = wz$

8. $z \cdot 1 = z$

9. $z \cdot 0 = 0$

10. $zwu = z(wu) = (zw)u$

11. $z(w + u) = zw + zu$

12. $z(w - u) = zw - zu$
The canonical form

**Theorem**

Let $z = (x, y)$ be a complex number, with $x$ and $y$ real. Then

$$z = x + iy.$$ 

**Proof:**

$$x + iy =$$

**Definition**

The form $x + iy$ is called the **canonical form** of $z$.

Henceforth we will always write complex numbers in canonical form.
Let \( z = x + iy \) and \( w = u + iv \) be two complex numbers, with \( x, y, u \) and \( v \) real. Then

\[
z + w =
\]

For the product of \( z \) and \( w \) we have

\[
z w =
\]
**Definition**

Let \( z = x + i \, y \) be a complex number with \( x \) and \( y \) real. Then \( x \) is the **real part of** \( z \) and \( y \) is the **imaginary part of** \( z \). We denote

\[
x = \text{Re} \, z \quad \text{and} \quad y = \text{Im} \, z.
\]
**Definition**

Let $z = x + iy$ be a complex number with $x$ and $y$ real. Then the **conjugate of** $z$ is the complex number $\bar{z}$ defined by

$$\bar{z} = x - iy.$$

- The conjugate of $z$ is the reflection of $z$ across the real axis.

⚠️ Notebook: Conjugate.nb
**Definition**

Let $z = x + i y$ be a complex number with $x$ and $y$ real. Then the **absolute value of** $z$ **is the distance of** $z$ **to 0:**

$$|z| = \sqrt{x^2 + y^2}.$$ 

- The definition is based on the Pythagorean theorem.
- The absolute value is sometimes called **modulus** or **norm.**
Let $z$ and $w$ be complex numbers, then

1. $z + w = \overline{z} + \overline{w}$
2. $z - w = \overline{z} - \overline{w}$
3. $z \overline{w} = \overline{z} \overline{w}$
4. $|z|^2 = z \overline{z}$
5. $|z \overline{w}| = |z| \ |w|$
6. $|z + w| \leq |z| + |w|$

Property 6 is called the **triangular inequality**.
The reciproke

Problem

For arbitrary $z \neq 0$, find a complex number $w$ such that $zw = 1$.

- Assume that $zw = 1$, then

$$
\bar{z}zw = \bar{z} \quad \Rightarrow \quad |z|^2 w = \bar{z} \quad \Rightarrow \quad \frac{1}{z} = w = \frac{1}{|z|^2} \bar{z}
$$

- The number $w$ is called the **reciproke of** $z$ and is denoted as $\frac{1}{z}$.

- The reciproke of $z$ is sometimes denoted as $z^{-1}$.

- Canonical form: if $z = x + iy$ with $x$ and $y$ real, then

$$
\frac{1}{z} = \frac{1}{|z|^2} \bar{z} = \frac{1}{x^2 + y^2} (x - iy) = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i.
$$
Definition

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$\frac{u}{z} = u \cdot \frac{1}{z}.$$ 

Notebook: Quotient.nb

- Equivalently we can write $\frac{u}{z} = \frac{1}{|z|^2} u \bar{z}$.

- Practical approach: multiply numerator and denominator with $\bar{z}$:

$$\frac{u}{z} = \frac{u \bar{z}}{z \bar{z}},$$

and elaborate $u \bar{z}$.

- Example:

$$\frac{3 + i}{1 + 2i} =$$
The argument of a complex number \( z \neq 0 \) is the angle that the line through 0 and \( z \) makes with the positive real axis. The argument of \( z \) is denoted as \( \arg(z) \).

The argument of 0 is not defined.

The argument is expressed in radians.

The argument is measured from the positive real axis.

If the direction is counter-clockwise, the argument is positive.

If the direction is clockwise, the argument is negative.

The argument is determined up to a multiple of \( 2\pi \).
The Euler function is the function that assigns to every real number \( \varphi \) the complex number

\[
e^{i\varphi} = \cos \varphi + i \sin \varphi.
\]

- The number \( e^{i\varphi} \) lies on the unit circle: \( |e^{i\varphi}| = 1 \).
- The real part of \( e^{i\varphi} \) is \( \cos \varphi \), the imaginary part of \( e^{i\varphi} \) is \( \sin \varphi \).
- The complex number \( e^{i\varphi} \) is the number on the unit circle with argument \( \varphi \).
Theorem

For every real number $\varphi$ and $\psi$ we have

$$e^{i(\varphi + \psi)} = e^{i\varphi}e^{i\psi}$$

- Use trigonometry formulas to derive

$$e^{i(\varphi + \psi)} =$$

- Expand the right-hand side:

$$e^{i\varphi}e^{i\psi} =$$
**Theorem**

Every complex number $z \neq 0$ can be written as the product of a positive real number and an Euler function value. In particular, if $r = |z|$ and $\varphi = \arg z$, then

$$z = r \, e^{i \varphi}$$

- Write $z = x + iy$ with $x$ and $y$ real, then

  $$\cos \varphi = \frac{x}{r} \quad \text{and} \quad \sin \varphi = \frac{y}{r}.$$  

- $z = x + iy$
  
  $$= r \cos \varphi + i(r \sin \varphi)$$
  
  $$= r(\cos \varphi + i \sin \varphi) = re^{i\varphi}.$$
In this part of the lecture we write $j$ in stead of $i$. 
<table>
<thead>
<tr>
<th>Component</th>
<th>Relation $v(t)$ vs. $i(t)$</th>
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<td>$v(t) = Ri(t)$</td>
<td>Dissipates energy</td>
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<td>$v(t) = \frac{1}{C} \int_0^t i(\tau) , d\tau$</td>
<td>Stores energy in an electric field</td>
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<td>Inductor</td>
<td>$v(t) = Li'(t)$</td>
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If you know $i(t)$, then $v(t)$ can be uniquely determined.

The component can therefore be regarded to be a system:

$$i(t) \xrightarrow{S} v(t)$$

or abbreviated: $i(t) \leftrightarrow v(t)$.

**Example:** for an inductor with inductance $L$ we have

$$i(t) \leftrightarrow Li'(t).$$
Definition

Let $S$ be a system. Let $x(t)$ and $y(t)$ be real signals for which $S$ has the following responses:

$$x(t) \mapsto u(t) \quad \text{and} \quad y(t) \mapsto v(t).$$

Then the response of $S$ to the input $x(t) + jy(t)$ is defined as

$$u(t) + jv(t).$$

Example: for an inductor with inductance $L$ we have

$$\cos(\omega t) \mapsto -\omega L \sin(\omega t) \quad \text{and} \quad \sin(\omega t) \mapsto \omega L \cos(\omega t),$$

hence

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \mapsto -\omega L \sin(\omega t) + j\omega L \cos(\omega t)$$

$$= j\omega L \left( \cos(\omega t) + j\sin(\omega t) \right)$$

$$= j\omega L e^{j\omega t}.$$
The transfer function

**Theorem**

For all systems corresponding to passive components, there exists a function $Z(\omega)$ such that

\[
e^{j\omega t} \mapsto Z(\omega) e^{j\omega t}
\]

- The function $Z(\omega)$ is called the **transfer function**.
- The transfer function does not depend on time, but can depend on $\omega$.
- For passive components, where the input is the current $i(t)$ through the component, and the response is the voltage $v(t)$ over the component, the function $Z(\omega)$ is called the **impedance** of the component, usually denoted as $Z$.
- **Example**: for an inductor with inductance $L$ we have

\[
e^{j\omega t} \mapsto j\omega L \, e^{j\omega t},
\]

so the impedance is $Z = j\omega L$. 

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Example

Let \( v(t) = 5 \cos(2\pi f t) \), where the frequency is equal to \( f = 10 \text{ kHz} \). The inductance \( L \) is 50 mH. Describe the current \( i(t) \) through the inductor as a function of \( t \). What is the amplitude of \( i(t) \)?

- The impedance of \( L \) is \( Z = j\omega L \), where \( \omega = 2\pi f \).
- Define \( \hat{v}(t) = 5e^{j\omega t} \), then

\[
\hat{v}(t) =
\]

Hence \( i(t) = \)

- The amplitude of \( i(t) \) is